

# A STUDY OF UNIFIED FINITE INTEGRAL INVOLVING THE EXTENDED JACOBI POLYNOMIAL, THE GENERALIZED POLYNOMIALS AND THE MULTIVARIABLE H- FUNCTION

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**ABSTRACT:** In this paper we establish finite integral which involve the product of the extended Jacobi polynomial, the Generalized Polynomials and the Multivariable H-function on account of the general nature of the function and polynomials occurring in these integral, my findings provide interesting unifications of a large number of (new and known) results.

**Keywords:** Beta function, H- function, Hermite Polynomial, Hyper Geometric function, Jacobi polynomials

## INTRODUCTION

To unify the classical orthogonal polynomials Viz. Jacobi, Hermite and Laguerre Fujiwara[2] defined a class of generalized classical polynomials by means of following Rodrigues formula:

$$R_n(x) = \frac{(-1)^n K^n}{n!(x-p)^\beta(q-x)^\alpha} \frac{d^n}{dx^n} [(x-p)^{\beta+n} (q-x)^{\alpha+n}], p < x < q, \alpha > -1, \beta > -1 \quad (1.1)$$

Denote these polynomials by  $F_n(\beta, \alpha; x)$  and call them extended Jacobi polynomials Thakare [9] obtained the following form of  $R_n(x) = F_n(\beta, \alpha; x)$

$$F_n(\beta, \alpha; x) = \frac{(-1)^n K^n (q-x)^n (1+\beta)_n}{n!} {}_2F_1 \left[ \begin{matrix} -n, -n-\alpha; \\ 1+\beta; \end{matrix} \middle| \frac{p-x}{q-x} \right] p < x < q \quad (1.2)$$

Fujiwara [2] proved that when  $p=-1, q=1$  and  $k=\frac{1}{2}$

$$F_n(\beta, \alpha; x) = P_n^{(\alpha, \beta)}(x)$$

Where  $P_n^{(\alpha, \beta)}(x) = \frac{(1+\beta)_n}{n!} \left(\frac{x-1}{2}\right)^n {}_2F_1 \left[ \begin{matrix} -n, -n-\alpha; \\ 1+\beta; \end{matrix} \middle| \frac{x+1}{x-1} \right]$  is Jacobi polynomial [4] (1.3)

The multivariable H- function occurring in the paper will be defined and represented in the following form [6,pp,251-252,eqn.(C.1)-(C.3)]

$$H \left[ \begin{matrix} Z_1 \\ \vdots \\ Z_r \end{matrix} \right] = H \left[ \begin{matrix} o, n : m_1, n_1; \dots; m_r, n_r \\ p, q : p_1, q_1; \dots; p_r, q_r \end{matrix} \middle| \begin{matrix} Z_1 \\ \vdots \\ Z_r \end{matrix} \right] \left[ \begin{matrix} (a_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)}) 1, p; \\ (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)}) 1, q; \\ (c_j^{(1)}, \gamma_j^{(1)}) 1, p_1; \dots; (c_j^{(r)}, \gamma_j^{(r)}) 1, p_r \\ (d_j^{(1)}, \delta_j^{(1)}) 1, q_1; \dots; (d_j^{(r)}, \delta_j^{(r)}) 1, q_r \end{matrix} \right]$$

$$= \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \psi(\varepsilon_1, \dots, \varepsilon_r) \phi_1(\varepsilon_1) \dots \phi_r(\varepsilon_r) z_1^{\varepsilon_1} \dots z_r^{\varepsilon_r} d\varepsilon_1 \dots d\varepsilon_r \quad (1.4)$$

For the convergence conditions of the integral given by (1.4) and other details of the multivariable H- function we refer to the book by Srivastava et al . [6,pp,252-253,eqns.(C.4)-(C.8)]

The generalized polynomials  $S_{N_1, \dots, N_s}^{M_1, \dots, M_s} \left[ \begin{matrix} X_1 \\ \vdots \\ X_s \end{matrix} \right]$  occurring herein will be defined and represented in the following form which differs slightly from that given by Srivastava [7, p. 185, eqn. (7)].

$$S_{N_1, \dots, N_s}^{M_1, \dots, M_s} \begin{bmatrix} x_1 \\ \vdots \\ x_s \end{bmatrix} = S_{N_1, \dots, N_s}^{M_1, \dots, M_s} [x_1, \dots, x_s]$$

$$= \sum_{k_1=0}^{[N_1/M_1]} \dots \sum_{k_s=0}^{[N_s/M_s]} \frac{(-N_1)_{M_1 k_1}}{k_1!} \dots \frac{(-N_s)_{M_s k_s}}{k_s!} A[N_1, k_1; \dots; N_s, k_s] x_1^{k_1} \dots x_s^{k_s} \quad (1.5)$$

Where  $N_i = 0, 1, 2, \dots; M_i \neq 0 [i = 1, \dots, s]$ .

$M_i$  is an arbitrary positive integer and the coefficients  $A[N_1, k_1; \dots; N_s, k_s]$  are arbitrary constants, real or complex.

If we take  $s = 1$  in the equation (1.5) and denote  $A[N, k]$  thus obtained by  $A_{N,k}$ , we arrive at the well know general class of polynomials  $S_N^M [x]$  introduced by Srivastava [8, p. 1, eqn. (1)].

**PRELIMINARIES**

In this paper we need the following results:

(i) [[1], p.10, eq.(13) Viz]  $\int_b^a (t - b)^{x-1} (a - t)^{y-1} dt = (a - b)^{x+y-1} B(x, y)$ ,  $Re(x) > 0, Re(y) > 0, b < a$  (2.1)

Where  $B(x,y)$  is beta function .

(ii) The Hyper Geometric function [4]  
 ${}_2F_1(a, b; c; z)$

$$= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n z^n}{(c)_n n!} \quad (2.2)$$

**MAIN INTEGRAL**

$$\int_p^q (x - p)^t (q - x)^\alpha F_n(\beta, \alpha; x) S_{N_1, \dots, N_s}^{M_1, \dots, M_s} \begin{bmatrix} e_1(x - p)^{u_1} (q - x)^{v_1} \\ \vdots \\ e_s(x - p)^{u_s} (q - x)^{v_s} \end{bmatrix} H \begin{bmatrix} z_1(x - p)^{u_1} (q - x)^{v_1} \\ \vdots \\ z_r(x - p)^{u_r} (q - x)^{v_r} \end{bmatrix} dx$$

$$= \frac{(-1)^n K^n (1 + \beta) n (q - p)^{t+\alpha+n+1}}{n!} \sum_{\ell=0}^{\infty} \frac{(-1)^\ell (-n - \alpha)_\ell}{\ell! (1 + \beta)_\ell}$$

$$\sum_{k_1=0}^{[N_1/M_1]} \dots \sum_{k_s=0}^{[N_s/M_s]} A[N_1, k_1; \dots; N_s, k_s] \prod_{j=1}^s \left( \frac{(-N_j)_{M_j k_j} e_j^{k_j}}{k_j!} (q - p)^{(u_j+v_j)} \right)$$

$$H \begin{matrix} 0, n + 2 & : & * & \left[ \begin{matrix} z_1(q - p)^{u_1+v_1} \\ \vdots \\ \vdots \\ z_r(q - p)^{u_r+v_r} \end{matrix} \right] \\ p + 2, q + 1 & : & * & \left[ \begin{matrix} A & : & * \\ B & : & * \end{matrix} \right] \end{matrix}$$

Where

$$A = (-t - \ell - u_1 k_1 \dots - u_s k_s; u_1, \dots, u_r) (-\alpha - n + \ell - v_1 k_1 \dots - v_s k_s; v_1, \dots, v_r) (\alpha_j; \alpha_j^{(1)}, \dots, \alpha_j^{(r)}) 1, p$$

$$B = (b_j; \beta_j^{(1)}, \dots, \beta_j^{(r)})_{1,q} (-\alpha - n - t - 1 - (u_1 + v_1) k_1 - \dots - (u_s + v_s) k_s; u_1 + v_1, \dots, u_r + v_r) \quad (3.1)$$

Also the asterisk (\*) occurring in the right hand side of (3.1) indicates that parameters at these places are the same as the multivariable H-function in (1.4).

The integral (3.1) Is valid under the following conditions :

(i)  $(u_i, v_i, u_i', v_i') \geq 0 \quad i = 1, \dots, r \text{ and } j=1, \dots, s$  (not all zero simultaneously )

$$(ii) \quad \operatorname{Re}(t+\ell)+\sum_{i=1}^r u_i \min_{1 \leq j \leq m_i} \left[ \operatorname{Re} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > -1$$

$$(iii) \quad \operatorname{Re}(\alpha+n-\ell)+\sum_{i=1}^r v_i \min_{1 \leq j \leq m_i} \left[ \operatorname{Re} \left( \frac{d_j^{(i)}}{\delta_j^{(i)}} \right) \right] > -1$$

**Proof of (3.1)**

To establish (3.1) replace the multivariable H- function by its Mellin-Barnes contour integral from. Now we interchange the order of x and  $\varepsilon_1, \dots, \varepsilon_r$  integrals (which is permissible under the conditions stated with (3.1) in the result thus obtained and get after a little simplification the left hand side of (3.1)) (say  $\Delta$ ) as:

$$\Delta = \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \phi_1(\varepsilon_1) \dots \phi_r(\varepsilon_r) \psi(\varepsilon_1 \dots \varepsilon_r) z_1^{\varepsilon_1} \dots z_r^{\varepsilon_r}$$

$$\left\{ \int_p^q (x-p)^{t+u_1\varepsilon_1+u_r\varepsilon_r} (q-x)^{\alpha+v_1\varepsilon_1+\dots+v_r\varepsilon_r} \right.$$

$$\left. \operatorname{Fn}(\beta, \alpha; x) S_{N_1, \dots, N_s}^{M_1, \dots, M_s} \left[ \begin{matrix} e_1(x-p)^{u_1}(q-x)^{v_1} \\ \vdots \\ e_s(x-p)^{u_s}(q-x)^{v_s} \end{matrix} \right] dx \right\} d\varepsilon_1 \dots d\varepsilon_r \quad (3.4)$$

Now in the inner integral (3.4) put the value of extended Jacobi polynomial  $\operatorname{Fn}(\beta, \alpha; x)$  from (1.2) and generalized polynomial [1.5], interchange the order of integration and summation (which is permissible under the condition stated with (3.1)). The equation (3.4) takes the following form after a little simplification with the help of known result (2.1)

$$\Delta = \frac{(-1)^n K^n (1+\beta)_n (q-p)^{t+\alpha+n+1}}{n!} \sum_{l=0}^{\infty} \frac{(-1)^l (-n-\alpha)_l}{l! (1+\beta)_l}$$

$$\sum_{k_1=0}^{[N_1/M_1]} \dots \sum_{k_s=0}^{[N_s/M_s]} A[N_1, k_1; \dots; N_s, k_s] \prod_{j=1}^s \left( \frac{(-N_j)_{M_j k_j} e_j^{k_j}}{k_j!} (q-p)^{(u_j+v_j)} \right)$$

$$\left\{ \frac{1}{(2\pi\omega)^r} \int_{L_1} \dots \int_{L_r} \phi_1(\varepsilon_1) \dots \phi_r(\varepsilon_r) \psi(\varepsilon_1 \dots \varepsilon_r) z_1^{\varepsilon_1} \dots z_r^{\varepsilon_r} (q-p)^{\sum_{i=1}^r (u_i+v_i)\varepsilon_i} \frac{[(t+l+\sum_{j=1}^s u_j k_j + \sum_{i=1}^r u_i \varepsilon_i + 1)(\alpha+n-l+\sum_{j=1}^s u_j k_j + \sum_{i=1}^r v_i \varepsilon_i + 1)]}{[(2+\alpha+n+t+\sum_{i=1}^r (u_i+v_i)\varepsilon_i + \sum_{j=1}^s (u_j+v_j)k_j]} \right\} d\varepsilon_1 \dots d\varepsilon_r \quad (3.5)$$

Finally on reinterpreting the multiple Mellin-Barnes contour integral occurring in right hand side of (3.5) in terms of the multivariable H-function We arrived at the desired result (3.1).

Special case: If we take  $p=-1, q=1$  and  $k=\frac{1}{2}$  In (3.1) we get the result which is same as obtained by Gupta and Pawan [3, p.84, eqn.(2.7)].

Further if we take  $s=1$  in (3.1) and further put  $N_1 = 0$  in the general class of polynomial thus obtained we arrive at the result Saxena and Ramawat [5, p.158, eqn.(2.4)].

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